The Minimality of the Superposition Axioms

Modal 6 — Explained in the Style of a Chess Journalist

Paul Koop

Preliminary Note: How to Read This File

This file explains **Modal 6**: the question of which of the five superposition axioms ASP1–ASP5 are really necessary for the derivation of the Trinity — and which are redundant.

It is written in the **same style** as the previous files: move by move, with symbol explanation at every step, with plain language for every statement.

What is presupposed: The first file (`modal5_schachjournalist_en.md`) should have been read. In particular, the tableau rules and the superposition axioms ASP1–ASP5 are presupposed as known.

What this file achieves: It shows two things:

1. **ASP1 is redundant** — it follows from ASP2 in S5. (This is a derivation proof.)
2. **ASP3, ASP4, ASP5 are indispensable** — for each of the three there is a countermodel in which the other axioms hold, but the respective axiom is violated, and in which the Trinity is false. (These are three countermodel constructions.)

At the end stands the **relativity of minimality**: There are two minimal subsets — one for the target statement, one for the superposition intuition.

Part I — The Question

1. What Does "Minimal" Mean?

An axiom set is **minimal for a target statement** if:

```
(1)  It derives the target statement.          (Sufficiency)
(2)  No proper subset derives it.              (Indispensability)
```

Symbol explanation:

- $\vdash$: "is derivable".
- $\subsetneq$: "proper subset of".
- $\nvdash$: "is not derivable".

In plain language: An axiom set is minimal for a target statement if it derives the target statement **and** if one cannot omit a single axiom without the derivation collapsing.

That is the definition with which Modal 6 works. It is not about "minimality" in an absolute sense, but about minimality **relative to a target statement**.

2. The Five Superposition Axioms

```
ASP1   C @w_super
ASP2   ∀v (w_super R v → (C @v ↔ v = w_super))
ASP3   ∀x Tr(x) @w_super
ASP4   ∀v (w_super R v → ∀x Tr(x) @v)
ASP5   ∀v (v ≠ w_super → w_super R v)
```

Symbol explanation (repetition):

- `C`: consciousness.
- `Tr`: Trinity.
- `w_super`: the distinguished world (superposition).
- `R`: the accessibility relation.
- `@w_super`, `@v`: world indices.
- `∀v`: "for all worlds v".
- `∀x`: "for all individuals x".
- `→`: "if … then …".
- `↔`: "if and only if".
- `=`: identity.
- `≠`: negation of identity.

In plain language:

- **ASP1:** Consciousness holds in w_super.
- **ASP2:** Consciousness holds only in w_super.
- **ASP3:** In w_super, the Trinity holds for all x.
- **ASP4:** What holds in w_super holds in all accessible worlds.
- **ASP5:** Every other world is accessible from w_super.

3. The Question of Modal 6

The question: Which of the five axioms are indispensable for the derivation of `□∀x Tr(x)`? Which are redundant?

The answer (preview):

```
ASP1 is redundant.              It follows from ASP2 in S5.
ASP2 is for the target statement
     not necessary, but for the
```

superposition intuition essential.
ASP3, ASP4, ASP5 are indispensable for the target statement.

The minimal subset for the target statement: {ASP3, ASP4, ASP5}. **The minimal subset for the superposition intuition:** {ASP1, ..., ASP5}.

Part II — The Redundancy Proof for ASP1

1. The Claim

Claim: ASP1 is derivable from ASP2 in S5.

Formally:

{ASP2} ⊢ ASP1 (in S5)

In plain language: If one has ASP2 as an axiom, one can derive ASP1 from it. ASP1 is therefore no additional assumption — it is already contained in ASP2.

2. The Proof Move by Move

Move 1 — Write down ASP2

```
BOARD       (empty — the proof begins)
MOVE        Insert ASP2 as axiom
SYMBOLS     ASP2 := ∀v (w_super R v → (C @v ↔ v = w_super))
WHY         ASP2 is available as an axiom. We write it down to
            instantiate it later.
PLAIN       "For every world v: If v is accessible from w_super,
            then consciousness holds in v if and only if v is equal
            to w_super."
NEW         Line 1: ∀v (w_super R v → (C @v ↔ v = w_super))
```

Move 2 — Apply ∀-rule

```
BOARD       Line 1: ∀v (w_super R v → (C @v ↔ v = w_super))
MOVE        Apply ∀-rule with v := w_super
SYMBOLS     ∀-rule: ∀v A → A[w_super/v]
            [w_super/v] = "replace v by w_super"
WHY         The formula has the form ∀v A. The ∀-rule allows
            substituting any individual for v — here w_super itself.
PLAIN       "We substitute w_super itself for v. Then:
            If w_super is accessible from w_super, then consciousness
            holds in w_super if and only if w_super is equal to
            w_super."
NEW         Line 2: w_super R w_super → (C @w_super ↔ w_super = w_super)
```

Move 3 — Reflexivity of R in S5

```
BOARD       Line 2: w_super R w_super → (C @w_super ↔ w_super = w_super)
MOVE        Apply reflexivity of R in S5
SYMBOLS     In S5, R is reflexive: ∀w (w R w)
WHY         In S5, the reflexivity of the accessibility relation holds.
            So in particular w_super R w_super.
PLAIN       "Every world is accessible from itself — so also w_super
            from w_super."
NEW         Line 3: w_super R w_super
```

Move 4 — Modus Ponens

```
BOARD       Line 2: w_super R w_super → (C @w_super ↔ w_super = w_super)
            Line 3: w_super R w_super
MOVE        Apply Modus Ponens
SYMBOLS     from A and A → B follows B
WHY         We have A (Line 3) and A → B (Line 2). So B follows.
PLAIN       "So: Consciousness in w_super if and only if w_super is
            equal to w_super."
NEW         Line 4: C @w_super ↔ w_super = w_super
```

Move 5 — Identity

```
BOARD       Line 4: C @w_super ↔ w_super = w_super
MOVE        Apply identity
SYMBOLS     w_super = w_super is always true (identity)
WHY         Every thing is identical with itself. So w_super = w_super
            is true.
PLAIN       "w_super is equal to w_super — that is always true."
NEW         Line 5: w_super = w_super
```

Move 6 — Resolve equivalence

```
BOARD       Line 4: C @w_super ↔ w_super = w_super
            Line 5: w_super = w_super
MOVE        Resolve equivalence
SYMBOLS     from A ↔ B and B follows A
WHY         We have A ↔ B (Line 4) and B (Line 5). So A follows.
PLAIN       "So: Consciousness in w_super."
NEW         Line 6: C @w_super
```

Move 7 — Result

```
BOARD       Line 6: C @w_super
MOVE        Note: That is ASP1
SYMBOLS     ASP1 := C @w_super
WHY         Line 6 is exactly ASP1.
PLAIN       "We have derived ASP1 — from ASP2 alone."
NEW         ASP1 is derived.
```

Final Result

```
FORMAL      {ASP2} ⊢ ASP1    (in S5)
PLAIN       "ASP1 is derivable from ASP2. ASP1 is therefore redundant
            – it is already contained in ASP2."
```

3. What the Proof Shows

What the proof shows:

- ASP1 is **no additional assumption**. It follows from ASP2.
- In the full axiomatics `{ASP1, ..., ASP5}`, ASP1 is **redundant**.
- One can **omit** ASP1 without losing derivational strength.

What the proof does not show:

- That ASP1 is **false**. It is true — but derivable.
- That ASP1 is **superfluous** for the superposition intuition. It is useful for the clarity of the intuition, even though it is formally redundant.

The metalogical point: Redundancy is **not a defect**. It is a **property** of an axiom set. One can retain redundant axioms if they clarify the intuition — or omit them if one wants minimal axiom systems.

Part III — The Indispensability of ASP3, ASP4, ASP5

1. The Strategy

The strategy: For each of the three axioms ASP3, ASP4, ASP5, we construct a **countermodel** — a model in which:

```
(1)  All other axioms hold.
(2)  The respective axiom is violated.
(3)  The target statement □∀x Tr(x) is false.
```

In plain language: A countermodel is a **concrete situation** in which the other axioms hold, but the respective axiom does not — and in which the Trinity is **not** necessary. If such a model exists, then one **cannot omit** the respective axiom — the target statement would then no longer be derivable.

The three countermodels:

```
Countermodel 1:  ASP3 violated, ASP4 and ASP5 hold.
Countermodel 2:  ASP4 violated, ASP3 and ASP5 hold.
Countermodel 3:  ASP5 violated, ASP3 and ASP4 hold.
```

2. Countermodel 1 — ASP3 Violated

The idea: There is a world `w_super` and a world `w*`. In `w_super`, the Trinity does **not** hold for all x (ASP3 violated). But `w*` is accessible from `w_super`, and in `w*` the Trinity holds for all x (ASP4 satisfied). And every other world is accessible from `w_super` (ASP5 satisfied).

The model in formulas:

```
W = {w_super, w*}
R = W × W (universal)
I:
  Tr(a) @w_super is FALSE for an individual a.
  Tr(x) @w* is TRUE for all x.
```

Symbol explanation:

- `W`: the set of worlds.
- `R`: the accessibility relation.
- `I`: the interpretation function.
- `W × W`: the set of all pairs of worlds (universal relation).
- `Tr(a) @w_super is FALSE`: In w_super, the Trinity does not hold for a.
- `Tr(x) @w* is TRUE`: In w*, the Trinity holds for all x.

Verification of the axioms:

```
ASP3:  ∀x Tr(x) @w_super     →  VIOLATED (because Tr(a) @w_super is false)
ASP4:  ∀v (w_super R v → ∀x Tr(x) @v)
       →  SATISFIED (because the only accessible worlds are w_super
          and w*. In w*, ∀x Tr(x) holds. In w_super it does not –
          but ASP4 only says: "What holds in w_super holds in all
          accessible worlds." Since ∀x Tr(x) does not hold in w_super,
          the implication is vacuously satisfied.)
ASP5:  ∀v (v ≠ w_super → w_super R v)
       →  SATISFIED (because R is universal, so every world is accessible)
```

Verification of the target statement:

```
□∀x Tr(x) @w_super is FALSE (because Tr(a) @w_super is false).
So ¬□∀x Tr(x) holds in the model.
```

Result: Without ASP3, the target statement is not derivable. **ASP3 is indispensable.**

3. Countermodel 2 — ASP4 Violated

The idea: There is a world `w_super` and a world `w1`. In `w_super`, the Trinity holds for all x (ASP3 satisfied). But in `w1` it does **not** hold for all x, although `w1` is accessible from `w_super` (ASP4 violated). And every other world is accessible from `w_super` (ASP5 satisfied).

The model in formulas:

```
W = {w_super, w1}
R = W × W (universal)
I:
  Tr(x) @w_super is TRUE for all x.
  Tr(a) @w1 is FALSE for an individual a.
```

Verification of the axioms:

```
ASP3:  ∀x Tr(x) @w_super     →  SATISFIED
ASP4:  ∀v (w_super R v → ∀x Tr(x) @v)
       →  VIOLATED (because w_super R w1, but ¬Tr(a) @w1)
ASP5:  ∀v (v ≠ w_super → w_super R v)
       →  SATISFIED (because R is universal)
```

Verification of the target statement:

```
□∀x Tr(x) @w1 is FALSE (because Tr(a) @w1 is false).
So ¬□∀x Tr(x) holds in the model.
```

Result: Without ASP4, the target statement is not derivable. **ASP4 is indispensable.**

4. Countermodel 3 — ASP5 Violated

The idea: There is a world `w_super` and a world `w1`. In `w_super`, the Trinity holds for all x (ASP3 satisfied). In `w1` it does **not** hold for all x. But `w1` is **not** accessible from `w_super` (ASP5 violated). The relation R is therefore **not** universal — `w_super` and `w1` are isolated.

The model in formulas:

```
W = {w_super, w1}
R = {(w_super, w_super), (w1, w1)}   (two isolated classes)
I:
  Tr(x) @w_super is TRUE for all x.
  Tr(a) @w1 is FALSE for an individual a.
```

Symbol explanation:

- `{(w_super, w_super), (w1, w1)}`: the relation contains only the reflexive pairs — each world is accessible only from itself.
- `two isolated classes`: w_super and w1 are separated from each other.

Verification of the axioms:

```
ASP3:  ∀x Tr(x) @w_super     →  SATISFIED
ASP4:  ∀v (w_super R v → ∀x Tr(x) @v)
       →  SATISFIED (because the only world accessible from w_super is
          w_super itself – and there ∀x Tr(x) holds)
ASP5:  ∀v (v ≠ w_super → w_super R v)
       →  VIOLATED (because w1 ≠ w_super, but w_super R w1 does not hold)
```

Verification of the target statement:

```
□∀x Tr(x) @w1 is FALSE (because Tr(a) @w1 is false).
So ¬□∀x Tr(x) holds in the model.
```

Result: Without ASP5, the target statement is not derivable. **ASP5 is indispensable.**

5. Summary of the Three Countermodels

```
Countermodel 1:  ASP3 violated, ASP4 and ASP5 hold.
                 → ASP3 is indispensable.
Countermodel 2:  ASP4 violated, ASP3 and ASP5 hold.
                 → ASP4 is indispensable.
Countermodel 3:  ASP5 violated, ASP3 and ASP4 hold.
                 → ASP5 is indispensable.
```

The point: For each of the three axioms there is a model in which the **other** axioms hold, but the respective axiom is violated — and in which the target statement is false. So one **cannot omit** any of the three.

Part IV — The Relativity of Minimality

1. Two Minimal Subsets

The minimal subset for the target statement:

```
{ASP3, ASP4, ASP5}
```

Justification: It derives □∀x Tr(x) (Modal 5), and no proper subset derives it (Part III of this file).

The minimal subset for the superposition intuition:

```
{ASP1, ASP2, ASP3, ASP4, ASP5}
```

Justification: It formalizes the full intuition (consciousness is the self-reflexive moment of the superposition). ASP1 is redundant but useful for clarity. ASP2 is not necessary for the target statement, but essential for the intuition.

2. The Relativity in Plain Language

The concept of "minimal" is relative. There is not **the** minimal axiom set — there are **two**, depending on what one wants to derive:

```
If one wants to derive only the Trinity:
  → {ASP3, ASP4, ASP5} is minimal.

If one wants to formalize the full superposition intuition:
  → {ASP1, ASP2, ASP3, ASP4, ASP5} is minimal (apart from the
    redundancy of ASP1).
```

That is no defect. It is an **insight**: The concept of minimality depends on the **goal**. Depending on the goal, a different axiom set is minimal.

3. What This Means for the Series

What remains:

- The Trinity is derivable from `{ASP3, ASP4, ASP5}`.
- The superposition intuition requires `{ASP1, ..., ASP5}`.
- ASP1 is redundant — it follows from ASP2.

What does not remain:

- The idea that there is **one** minimal axiom set.
- The idea that minimality is absolute.

The metalogical consequence: The analysis of minimality is **relative** — it depends on the goal. That is an **insight**, not a defect.

Afterword

You have read this file. You have followed the redundancy proof for ASP1 move by move — with symbol explanation at every step and plain language for every statement. You have seen the three countermodels for ASP3, ASP4, ASP5. You have understood the relativity of minimality.

What remains is the insight:

```
ASP1 is redundant.
ASP3, ASP4, ASP5 are indispensable.
Minimality is relative – it depends on the goal.
```

The next file (Modal 7) treats **non-uniqueness**: There are at least four systematically different extensions of S5 that all derive the Trinity. The Trinity is **axiom-invariant**.

This file was written in the spirit of strict modal logic, yet in the language of a chess journalist — for the proof requires both: the precision of the formula and the vividness of the move.