

The Trinity as Limit

Topological Modal Logic of the Monistic Structure

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Zusammenfassung

In modal5.md, the definitions $T := \{ x \mid U < x < S \}$, $U := \lim_{\leftarrow} \{x \rightarrow \inf\} T$, and $S := \lim_{\leftarrow} \{x \rightarrow \sup\} T$ appear. These definitions are informal. They use a metric limit concept that is not readily available in first-order modal logic. The present treatise provides a precise modal-logical version of this discourse. It extends S5 by a limit operator L with neighborhood semantics, defines $U := L_{\inf} T$ and $S := L_{\sup} T$, and proves: In S5+SP+L, the Trinitarian formula holds. The Trinity thus proves to be a limit structure, not a substantial object. The treatise refines the discourse of the limit, it does not replace it.

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<i>Past and future are horizons of knowledge.</i>	
<i>The present is the locus of reality.</i>	
<i>Nothing that was, is lost.</i>	

Preface: The Informal Limit Definition

In `modal5.md`, the Trinity is introduced as a limit structure. The central definitions are:

$$\begin{aligned}
 T &:= \{x \mid U < x < S\} \\
 U &:= \lim_{x \rightarrow \inf} T \\
 S &:= \lim_{x \rightarrow \sup} T \\
 Tr &:= U \wedge T \wedge S
 \end{aligned}$$

These definitions are **informal**. They use a **metric limit concept** that is not readily available in first-order modal logic. The expression $\lim_{x \rightarrow \inf} T$ presupposes a metric or at least an order topology on T — neither of which is self-evident in pure predicate logic with modal operators.

The present treatise provides a **precise modal-logical version** of this discourse. It extends S5 by a **limit operator** $\mathcal{L}$ with neighborhood semantics, defines U and S as limits in the sense of this operator, and proves: In S5+SP+ $\mathcal{L}$, the Trinitarian formula holds.

The central claim of this treatise is:

The Trinity was always thought as a limit structure. The treatise refines this discourse, it does not replace it. The limit concept is translated from a metric into a modal-logical language — without changing the substance of the original intuition.

The investigation remains faithful to the methodological self-understanding of `modal5.md`: It shows a **formalization**, not a new metaphysical thesis. What it shows is that the discourse of the limit can be precisely grasped in modal logic.

Topological Modal Logic: A Brief Introduction

Why Topology?

Classical modal logic S5 uses an **accessibility relation** R between possible worlds. In S5, this relation is an equivalence relation. It allows us to speak about necessity and possibility, but it does **not** allow us to speak about **convergence** or **limits**.

For the discourse of limits, we need a **topological structure**. Topological modal logic extends S5 by such a structure.

Neighborhood Semantics

Neighborhood semantics is a generalization of Kripke semantics. Instead of an accessibility relation R , it uses a **neighborhood function** N that assigns to each world w a set of sets of worlds:

$$N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$$

$N(w)$ is the set of **neighborhoods** of w . A neighborhood of w is a set of worlds that “surrounds” w .

Intuition: $N(w)$ describes which sets of worlds lie in the vicinity of w .

The Neighborhood Operator

The **neighborhood operator** $\mathcal{N}$ is defined by:

$$\mathcal{N}p@w \iff \{v \mid p@v\} \in N(w)$$

Reading: “ $\mathcal{N}p$ holds in w ” exactly when the set of worlds in which p holds is a neighborhood of w .

Properties:

- $\mathcal{N}$ is **not** normal — it does not distribute over implication.
- $\mathcal{N}$ allows **finer** distinctions than $\Box$.
- In S5: $\Box p \rightarrow \mathcal{N}p$, but not conversely.

Convergence

A **convergence structure** on W is a family of filters $\mathcal{F}_w$ for each world w . A **filter** is a set of subsets of W closed under finite intersections and supersets.

Intuition: $\mathcal{F}_w$ describes which sets of worlds lie “eventually” in the vicinity of w .

Convergence: A sequence of worlds (w_n) converges to w if for every neighborhood $A \in N(w)$, almost all w_n lie in A .

Limit Operators

A **limit operator** $\mathcal{L}$ is defined by:

$$\mathcal{L}p@w \iff \text{there is a sequence } (w_n) \rightarrow w \text{ with } p@w_n \text{ for almost all } n$$

Reading: “ $\mathcal{L}p$ holds in w ” exactly when p holds in almost all worlds of a sequence converging to w .

Important: $\mathcal{L}$ is **not** a normal operator. It does not satisfy the Kripke axiom K . Instead, it satisfies **convergence axioms**.

The Limit Operator L: Definition, Axioms, Tableau Rules

Definition

We extend the language of S5 by a **binary limit operator** $\mathcal{L}_{\text{inf}}$ and $\mathcal{L}_{\text{sup}}$:

- $\mathcal{L}_{\text{inf}}p$ means: “ p holds in the lower limit.”
- $\mathcal{L}_{\text{sup}}p$ means: “ p holds in the upper limit.”

Semantics:

$$\mathcal{L}_{\text{inf}}p@w \iff \text{there is a monotonically decreasing sequence } (w_n) \rightarrow w \text{ with } p@w_n$$

$$\mathcal{L}_{\text{sup}}p@w \iff \text{there is a monotonically increasing sequence } (w_n) \rightarrow w \text{ with } p@w_n$$

Axioms

$$\begin{aligned} L1 &:= \Box(\mathcal{L}_{\text{inf}}p \rightarrow \Diamond p) \\ L2 &:= \Box(\mathcal{L}_{\text{sup}}p \rightarrow \Diamond p) \\ L3 &:= \Box(\mathcal{L}_{\text{inf}}(p \rightarrow q) \rightarrow (\mathcal{L}_{\text{inf}}p \rightarrow \mathcal{L}_{\text{inf}}q)) \\ L4 &:= \Box(\mathcal{L}_{\text{sup}}(p \rightarrow q) \rightarrow (\mathcal{L}_{\text{sup}}p \rightarrow \mathcal{L}_{\text{sup}}q)) \\ L5 &:= \Box(\mathcal{L}_{\text{inf}}p \rightarrow \mathcal{N}p) \\ L6 &:= \Box(\mathcal{L}_{\text{sup}}p \rightarrow \mathcal{N}p) \\ L7 &:= \Box(\mathcal{N}p \rightarrow p) \end{aligned}$$

Explanation:

- L1, L2: Limits are possible.
- L3, L4: Limit operators distribute over implication.
- L5, L6: Limits imply neighborhood.
- L7: Neighborhood implies actuality.

Tableau Rules

$$\begin{array}{c}
(\neg\mathcal{L}_{\text{inf}}) \quad \frac{\neg\mathcal{L}_{\text{inf}} A}{\Diamond\neg A} \\
(\neg\mathcal{L}_{\text{sup}}) \quad \frac{\neg\mathcal{L}_{\text{sup}} A}{\Diamond\neg A} \\
(\mathcal{L}_{\text{inf}}) \quad \frac{\mathcal{L}_{\text{inf}} A}{\mathcal{N}A} \\
(\mathcal{L}_{\text{sup}}) \quad \frac{\mathcal{L}_{\text{sup}} A}{\mathcal{N}A} \\
(\mathcal{N}) \quad \frac{\mathcal{N}A}{A}
\end{array}$$

Important: The rules for $\mathcal{L}_{\text{inf}}$ and $\mathcal{L}_{\text{sup}}$ are **one-sided** — they only allow the transition from limit to neighborhood, not the other way.

Reconstruction of U and S

The Informal Definitions

In `modal5.md`:

$$\begin{aligned}
U &:= \lim_{x \rightarrow \text{inf}} T \\
S &:= \lim_{x \rightarrow \text{sup}} T
\end{aligned}$$

These definitions use a **metric** limit concept. We translate them into topological modal logic.

The Formal Definitions

$$\begin{aligned}
U &:= \mathcal{L}_{\text{inf}} T \\
S &:= \mathcal{L}_{\text{sup}} T
\end{aligned}$$

Explanation:

- U is the **lower limit** of Totality T . It is what comes closest to the “beginning.”
- S is the **upper limit** of Totality T . It is what comes closest to the “end.”

Important: U and S are **not elements** of T . They are limits in the sense of the operator $\mathcal{L}$.

Totality T

$$T := \{x \mid U < x < S\}$$

In modal-logical language:

$$T := \{x \mid \neg\mathcal{L}_{\text{inf}}\neg Tr(x) \wedge \neg\mathcal{L}_{\text{sup}}\neg Tr(x)\}$$

Explanation: T is the set of x for which: x lies “between” U and S , i.e., x is not itself a limit, but an element of the open interval.

The Trinity

$$\boxed{Tr := U \wedge T \wedge S}$$

Proof of the Trinity in S5+SP+L

The Proof

Theorem L-A: In S5+SP+ $\mathcal{L}$, $\Box \forall x Tr(x)$ holds.

Proof:

1. According to `modal5.md`, in S5+SP: $\Box \forall x Tr(x)$ — but with the informally defined Tr .
2. We must show that the formal definition of Tr with $\mathcal{L}$ yields the same structure.

Step 1: From ASP3: $\forall x Tr(x)@w_{\text{super}}$.

Step 2: From ASP4: For all v with $w_{\text{super}}Rv$, $\forall x Tr(x)@v$.

Step 3: We show that from $\forall x Tr(x)@w_{\text{super}}$ follows: $\mathcal{L}_{\text{inf}}T@w_{\text{super}}$ and $\mathcal{L}_{\text{sup}}T@w_{\text{super}}$.

- **Proof of $\mathcal{L}_{\text{inf}}T$:** Consider the monotonically decreasing sequence $w_{\text{super}} \supseteq w_1 \supseteq w_2 \supseteq \dots$, in which each w_i contains the Trinity in a smaller “section.” Since Tr holds in w_{super} and T is the open interval, T holds in almost all w_i . Hence $\mathcal{L}_{\text{inf}}T@w_{\text{super}}$.
- **Proof of $\mathcal{L}_{\text{sup}}T$:** Analogously with a monotonically increasing sequence.

Step 4: From L5, L6, and L7: $\mathcal{N}T@w_{\text{super}}$ and $T@w_{\text{super}}$.

Step 5: From Steps 3 and 4: $U \wedge T \wedge S@w_{\text{super}}$, i.e., $Tr@w_{\text{super}}$.

Step 6: Since w_{super} is accessible to all worlds (ASP5), Tr holds in all worlds. Hence $\Box \forall x Tr(x)$.

q.e.d.

The Structure of the Proof

The proof shows:

- The formal definition of U and S as limits is **compatible** with the informal definition.
- The Trinitarian formula follows in S5+SP+ $\mathcal{L}$ exactly as in S5+SP.
- The topological extension adds **no new metaphysical assumptions** — it only refines the language.

Comparison with the Original Definition

The Informal Definition

$$U := \lim_{x \rightarrow \text{inf}} T$$

What it provides:

- It gives an intuitive idea: U is the lower limit of T .
- It is mathematically intuitive.
- But it is **not formal** — it presupposes a metric.

The Formal Definition

$$U := \mathcal{L}_{\text{inf}} T$$

What it provides:

- It is formulated in a precise modal-logical language.
- It uses a limit operator $\mathcal{L}$ whose semantics are clearly defined.
- It allows the proof of the Trinitarian formula to be conducted **formally**.

What Remains the Same

The **structure** of the definition is the same:

- U is the lower limit of T .
- S is the upper limit of T .
- T is the open interval between U and S .

The **intuition** is the same:

- The Trinity is a limit structure.
- The three aspects are not separate substances, but perspectives on the same reality.

What Changes

The **language** changes:

- Instead of a metric limit, we use a topological limit.
- Instead of a metric, we use a neighborhood structure.
- Instead of a number line, we use a modal-logical frame.

The **rigor** changes:

- The proof is now conducted **formally**.
- The definitions are **precise**.
- The dependencies are **explicit**.

What Follows? The Trinity Is a Limit, Not an Object

The Central Insight

The Trinity is **not an object**. It is **not a thing** among things. It is a **limit** — what stands at the end of a sequence without itself being an element of the sequence.

Formally: U and S are not elements of T . They are limits in the sense of $\mathcal{L}$.

Metaphysically: The Trinity is not an additional entity. It is the **structure** of Totality itself.

Consequences

Consequence 1: The Trinity cannot be “found” like an object. It can only be **thought** as a limit.

Consequence 2: The Trinity is **not local**. It is not at a particular place or time. It is a **global structure**.

Consequence 3: The Trinity is **not substantial**. It is not a “being,” but a **form**. It is what remains when all substances are subtracted.

The Relation to Tradition

Classical Trinitarian doctrine speaks of “three persons” and “one essence.” The present treatise shows: This way of speaking can be **precisely** grasped — but not as a statement about substances, but as a statement about **limits**.

- “Three persons” corresponds to the three limits U, T, S .
- “One essence” corresponds to the one structure constituted by the three limits.

Open Questions: Which Topology Is the Right One?

The Question

The treatise has used **one** topology: neighborhood semantics with convergence structure. But there are **many** possible topologies:

- The **discrete** topology.
- The **metric** topology.
- The **order** topology.
- The **weak** topology.
- The **product** topology.

Which is the right one?

Possible Answers

Answer 1: Neighborhood semantics is the **weakest** topology that allows the discourse of limits. It is therefore the **minimal** choice.

Answer 2: The **order topology** is the natural topology for Totality T , because T already has an order.

Answer 3: The **weak** topology (also: weak convergence) is the right one, because it is the **weakest** topology in which the Trinitarian formula can be derived.

Answer 4: There is no “right” topology. The choice depends on metaphysical or physical preferences. We stand at the beginning of infinity.

What We Have Not Shown

We have **not** shown that neighborhood semantics is the only possible topology. We have only shown that it is **sufficient**.

The question of the **minimality** of the topology remains open. It is the subject of **Modal 9** and beyond.

Conclusion: What Has Been Shown?

The present investigation has shown:

1. **The informal limit definition from modal5.md can be translated into a precise modal-logical language.** The limit operator $\mathcal{L}$ with neighborhood semantics accomplishes this translation.
2. **In $S5+SP+\mathcal{L}$, the Trinitarian formula holds.** The topological extension is sufficient.
3. **The Trinity is a limit, not an object.** It is the structure of Totality itself, not an additional entity.
4. **The formal definition is compatible with the informal definition.** It does not replace it, it refines it.

The central claim of this treatise is:

The Trinity was always thought as a limit structure. The treatise refines this discourse, it does not replace it. The limit concept is translated from a metric into a modal-logical language — without changing the substance of the original intuition.

The question of which topology is the right one remains open. It is the subject of further investigations.

Appendix A: The Axioms in an Overview

Basic Axioms (from modal5.md)

$$\begin{aligned}
A1 &:= \Box \neg \exists x \exists y \textit{FundamentallySeparated}(x, y) \\
A4 &:= \Diamond \exists w \textit{World}(w) \\
A11 &:= \forall w (\textit{WorldSeparatedFromConsciousness}(w) \leftrightarrow \neg \exists c (\textit{Consciousness}(c) \wedge c(w))) \\
A12 &:= \forall p. \textit{Experienceable}(p) \leftrightarrow \exists w. \textit{Realized}(w, p) \\
A13 &:= \Box \forall x (T(x) \rightarrow U(x)) \\
A14 &:= \Box \forall x (U(x) \rightarrow S(x)) \\
A15 &:= \Box \forall x (S(x) \rightarrow T(x))
\end{aligned}$$

Superposition Axioms (from modal5.md)

$$\begin{aligned}
ASP1 &:= C@w_{\text{super}} \\
ASP2 &:= \forall v (w_{\text{super}} Rv \rightarrow (C@v \leftrightarrow v = w_{\text{super}})) \\
ASP3 &:= \forall x \textit{Tr}(x)@w_{\text{super}} \\
ASP4 &:= \forall v (w_{\text{super}} Rv \rightarrow \forall x \textit{Tr}(x)@v) \\
ASP5 &:= \forall v (v \neq w_{\text{super}} \rightarrow w_{\text{super}} Rv)
\end{aligned}$$

Limit Axioms (new)

$$\begin{aligned}
L1 &:= \Box(\mathcal{L}_{\text{inf}}p \rightarrow \Diamond p) \\
L2 &:= \Box(\mathcal{L}_{\text{sup}}p \rightarrow \Diamond p) \\
L3 &:= \Box(\mathcal{L}_{\text{inf}}(p \rightarrow q) \rightarrow (\mathcal{L}_{\text{inf}}p \rightarrow \mathcal{L}_{\text{inf}}q)) \\
L4 &:= \Box(\mathcal{L}_{\text{sup}}(p \rightarrow q) \rightarrow (\mathcal{L}_{\text{sup}}p \rightarrow \mathcal{L}_{\text{sup}}q)) \\
L5 &:= \Box(\mathcal{L}_{\text{inf}}p \rightarrow \mathcal{N}p) \\
L6 &:= \Box(\mathcal{L}_{\text{sup}}p \rightarrow \mathcal{N}p) \\
L7 &:= \Box(\mathcal{N}p \rightarrow p)
\end{aligned}$$

Definitions

$$\begin{aligned}
T &:= \{x \mid U < x < S\} \\
U &:= \mathcal{L}_{\text{inf}}T \\
S &:= \mathcal{L}_{\text{sup}}T \\
Tr &:= U \wedge T \wedge S
\end{aligned}$$

Appendix B: Comparison of Definitions

Aspect	Informal (modal5.md)	Formal (modal8.md)
Language	Metric	Modal-logical
Basic concept	Number line	Neighborhood semantics
Limit	$\lim_{x \rightarrow \text{inf}}$	L_{inf}
Rigor	Intuitive	Formal
Trinity	Structure	Limit structure

Afterword

You have read this treatise. Perhaps the refinement of the limit definition surprised you. Perhaps the idea of topological modal logic impressed you. Both are fine.

The investigation has shown: The Trinity was always thought as a limit structure. The treatise refines this discourse, it does not replace it. The limit concept is translated from a metric into a modal-logical language — without changing the substance of the original intuition.

What remains is the insight that the Trinity is **not an object**. It is a **limit** — what stands at the end of a sequence without itself being an element of the sequence. It is the **structure** of Totality itself.

One final question:

Do we dare the leap into openness to the limit — even though the limit itself is not formally reachable, but only describable in the language of topology?

This treatise was written in the spirit of rigorous modal logic, yet in the language of philosophy – for truth requires both: the precision of the formula and the breadth of the concept.